

30.10.18.

PREDAVANJE 6

data jna i oblast egzistencije \rightarrow napisati barem jedan Košijev zadatak upr.

Nehomogena lin. dif. jna

$$L(y) = y^{(n)} + a_1(x) \cdot y^{(n-1)} + \dots + a_n(x)y = b(x)$$

\rightarrow posmatramo jnu $L(y) = b(x)$
 $b(x) \neq 0$

T: Neka je $\varphi_H(x)$ opšte rješenje odgovarajuće homogene ^{dif.} jednačine. $L(y) = 0$, i neka je $\varphi_p(x)$ neko partikularno rješenje jne $L(y) = b(x)$.
Tada je opšte rješenje jne $L(y) = b(x)$: $\varphi_H + \varphi_p$

u homogenom rješenju imamo n const.

\rightarrow Sad je pitanje kako naći rješenje

METOD VARIJACIJE KONSTANTI

\rightarrow prvo tražimo homogeno rješenje:

$$L(y) = 0$$

$$y_H = c_1 \varphi_1(x) + \dots + c_n \varphi_n(x)$$

\rightarrow rješavamo $L(y) = b(x)$

\rightarrow pretpostavljamo da c nisu konstante već da zavise od x :

$$y_{\text{pr}} = c_1(x) \cdot \varphi_1 + \dots + c_n(x) \cdot \varphi_n$$

\rightarrow ovo rješenje ubacujemo u jnu $L(y) = b(x)$

$$y' = \underbrace{c_1'(x)\varphi_1(x) + \dots + c_n'(x)\varphi_n(x)} +$$

$$+ c_1(x)\varphi_1'(x) + \dots + c_n(x)\varphi_n'(x)$$

→ pretpostavimo $= 0$

$$y'' = \underbrace{c_1'(x)\varphi_1'(x) + \dots + c_n'(x)\varphi_n'(x)} +$$

$$+ c_1(x)\varphi_1''(x) + \dots + c_n(x)\varphi_n''(x)$$

→ pretpostavljamo da je ovo gore $= 0$

$$y^{(n-1)} = \underbrace{c_1'(x)\varphi_1^{(n-2)}(x) + \dots + c_n'(x)\varphi_n^{(n-2)}(x)} +$$

$$+ c_1(x)\varphi_1^{(n-1)}(x) + \dots + c_n(x)\varphi_n^{(n-1)}(x)$$

$$y^{(n)} = c_1'(x)\varphi_1^{(n-1)}(x) + \dots + c_n'(x)\varphi_n^{(n-1)}(x) +$$

$$+ c_1(x)\varphi_1^{(n)}(x) + \dots + c_n(x)\varphi_n^{(n)}(x)$$

→ ovdje ne smijemo pretpostaviti da je ovo gore $= 0$

→ vratimo se sa ovim u 1-juu

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = b(x)$$

$$\rightarrow L(y) = b(x) \quad \vdots$$

$$c_1'(x)\varphi_1^{(n-1)}(x) + \dots + c_n'(x)\varphi_n^{(n-1)}(x) +$$

$$+ c_1(x)\varphi_1^{(n)}(x) + \dots + c_n(x)\varphi_n^{(n)}(x) +$$

$$+ a_1(x) \left(c_1(x)\varphi_1^{(n-1)}(x) + \dots + c_n(x)\varphi_n^{(n-1)}(x) \right) +$$

$$\vdots + a_n(x) \left(c_1(x)\varphi_1^{(n-1)}(x) + \dots + c_n(x)\varphi_n^{(n-1)}(x) \right) = b(x)$$

$a_n \cdot y^{(n-1)}$

$a_n \cdot y$

~~$y^{(n)}$~~

$$c_1'(x) \cdot \varphi_1^{(n-1)}(x) + \dots + c_n'(x) \varphi_n^{(n-1)}(x) +$$

$$+ c_1(x) \left[\varphi_1^{(n)}(x) + a_1(x) \varphi_1^{(n-1)}(x) + \dots + a_n(x) \varphi_1(x) \right] +$$

$$\dots + c_n(x) \left(\varphi_n^{(n)}(x) + \dots + a_n(x) \varphi_n(x) \right)$$

$L(\varphi_1(x)) = 0$

$L(\varphi_n) = 0$

→ pretpostaviti da je $\varphi_1, \dots, \varphi_n$

ako je to rj; postavimo da je:

$$c_1'(x) \cdot \varphi_1^{(n-1)}(x) + \dots + c_n'(x) \cdot \varphi_n^{(n-1)}(x) = b(x)$$

→ postaviti uslove koje moraju da zadovoljavaju c_1, \dots, c_n

$$\begin{cases} c_1'(x) \varphi_1(x) + \dots + c_n'(x) \varphi_n(x) = 0 \\ c_1'(x) \varphi_1'(x) + \dots + c_n'(x) \varphi_n'(x) = 0 \\ \vdots \\ c_1'(x) \varphi_1^{(n-2)}(x) + \dots + c_n'(x) \varphi_n^{(n-2)}(x) = 0 \\ c_1'(x) \varphi_1^{(n-1)}(x) + \dots + c_n'(x) \varphi_n^{(n-1)}(x) = b(x) \end{cases}$$

da bi bilo rješenje c_1, \dots, c_n moraju ovo da zadovoljavaju

$$W(x) \neq 0 \Rightarrow \exists! \text{ rješ. sist.}$$

$$c_i' = \frac{W_i(x)}{W(x)} ; i = 1, \dots, n$$

→ Kramerovo pravilo → i-tu kolonu zamijenimo kolonom slob. članova

→ dobiti smo dif. j-nu sa razdv. pravij.

$$c_i(x) = \int \frac{w_i(x)}{w(x)} dx + r_i^0, \quad i=1, \dots, n$$

$$\begin{aligned}
 y &= c_1(x) \varphi_1(x) + \dots + c_n(x) \varphi_n(x) = \\
 &= \left(\int \frac{w_1(x)}{w(x)} dx + r_1^0 \right) \cdot \varphi_1(x) + \dots + \left(\int \frac{w_n(x)}{w(x)} dx + r_n^0 \right) \cdot \varphi_n(x) = \\
 &= \underbrace{r_1^0 \varphi_1(x) + \dots + r_n^0 \varphi_n(x)}_{y_h \text{ (1)}} + \underbrace{\varphi_1(x) \int \frac{w_1(x)}{w(x)} dx + \dots + \varphi_n(x) \int \frac{w_n(x)}{w(x)} dx}_{y_p}
 \end{aligned}$$

$$L(y) = P_m(x) \cdot e^{rx}$$

$$\begin{array}{l}
 \underbrace{\hspace{10em}}_{b(x)} \\
 \rightarrow y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y
 \end{array}$$

najčešće se radi o konst. coef.

→ kada je $b(x)$ ovog oblika → $b(x)$ je kvazipolinom

$L(y) = 0$; imamo polinom

$$P(\lambda) = 0 \rightarrow \lambda_1, \lambda_2, \dots, \lambda_n$$

$$y_h = \dots$$

y_p tražimo na sl. način:

ako je $r \in \{\lambda_1, \dots, \lambda_n\} \rightarrow s$ -višestrukost r kao korijena karak. polinoma.

tada je $y_p = x^s \cdot \underbrace{Q_m(x)}_{\text{ Samo ovo nepoznata}} \cdot e^{rx}$

→ ovo sad zamijenimo u J_m :

$$L(y) = P_m(x) \cdot e^{rx}$$

→ r u opštem slučaju $\in \mathbb{C}$

ali mi razdvajamo

$$r = \alpha + i\beta$$

$$b(x) = e^{\alpha x} (P_{1m}(x) \cos \beta x + P_{2l}(x) \sin \beta x)$$

$S \rightarrow$ višestrukost korigena $\alpha \pm i\beta$ kakak. polinomnog

$$y_p = x^s \cdot e^{\alpha x} (Q_{1k}(x) \cos \beta x + Q_{2k}(x) \sin \beta x)$$

$$k = \max \{m, l\}$$

$$L(y) = P_{1m_1}(x) \cdot e^{r_1 x} + \dots + P_{sm_s}(x) \cdot e^{r_s x}$$

$$y_p = y_{p1} + \dots + y_{ps}$$

$$y_{p1}: L(y) = P_{1m_1}(x) \cdot e^{r_1 x}$$

$$\vdots$$
$$L(y) = P_{sm_s}(x) \cdot e^{r_s x}$$

Primeri rehomogena dif. j. na sa konst. koef.:

$$y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0$$

$$y'' - 2y' + y = 0$$

$$P(\lambda) = \lambda^2 - 2\lambda + 1 = 0$$

$$\rightarrow (\lambda - 1)^2 = 0; \quad \lambda_1 = \lambda_2 = 1; \quad s = 1$$

$$\lambda = 1 \rightarrow r = 1 \rightarrow y_1(x) = e^x$$

$$y_2(x) = x e^x$$

$$y_{\text{hom}} = c_1 e^x + c_2 x e^x$$

part. rj. tražimo tako što pretpostavljamo:

$$\begin{cases} c_1'(x) \cdot e^x + c_2'(x) \cdot x e^x = 0 \\ c_1'(x) \cdot e^x + c_2'(x) \cdot e^x(x+1) = \frac{e^x}{x} \end{cases}$$

$$(1) \begin{cases} c_1'(x) + c_2'(x) \cdot x = 0 \\ c_1'(x) + c_2'(x)(x+1) = \frac{1}{x} \end{cases}$$

$$c_2'(x) = \frac{1}{x}$$

$$c_2(x) = \int \frac{1}{x} dx = \ln x + r_2$$

Zamjenimo u (1)

$$\rightarrow c_1'(x) = -x c_2'(x)$$

$$c_1'(x) = -1 \rightarrow c_1(x) = -x + r_1$$

$$y = (-x + r_1) e^x + (\ln x + r_2) x e^x$$

$$y = \underbrace{r_1 e^x + r_2 x e^x}_{y_H} - \underbrace{x e^x + x e^x \ln x}_{y_P}$$

Primer

$$y'' - 4y' + 8y = \underbrace{e^{2x}}_{\text{zbir}} + \underbrace{\sin 2x}_{\text{kvazipolinom}}$$

homogeno rješenje:

$$y'' - 4y' + 8y = 0; \quad P(\lambda) = \lambda^2 - 4\lambda + 8 = 0$$

$$\lambda_{1,2} = 2 \pm 2i$$

$$y_H = c_1 e^{2x} \cos 2x + c_2 e^{2x} \sin 2x = e^{2x} (c_1 \cos 2x + c_2 \sin 2x)$$

$$y_P = y_{P1} + y_{P2}$$

\rightarrow za part. rj. 1 \rightarrow rješavamo jnu:

$$y_{P1} \text{ : } y'' - 4y' + 8y = e^{2x}$$

$$2 \notin \{2 \pm 2i\} \Rightarrow S = 0$$

$$y_{P1} = x^0 e^{2x} \cdot a = a e^{2x}$$

\hookrightarrow polinom nultog stepena

konstantu dobijamo kad zamijenimo $y_{p1} \dots$

$$y_{p1}' = 2ae^{2x} ; y_{p1}'' = 4ae^{2x}$$

$$4ae^{2x} - 4 \cdot 2ae^{2x} + 8ae^{2x} = e^{2x} \quad | \cdot e^{-2x}$$

$$4a = 1 \rightarrow a = \frac{1}{4} \rightarrow$$

$$y_{p1} = \frac{e^{2x}}{4}$$

$$y_{p2}: y'' - 4y' + 8y = \sin 2x$$

polinom nultog stepena; $\cos 2x$ \rightarrow konst je 0

ali ne smijemo sin u rješt. za-
boraviti \rightarrow

$$\alpha + i\beta = 2i \neq \{2 \pm 2i\} \Rightarrow S = 0;$$

$$y_{p2} = (a \cos 2x + b \sin 2x) \cdot \underset{\downarrow}{x^0} \cdot \underset{\downarrow}{e^{0x}}$$

$$y_{p2} = a \cos 2x + b \sin 2x$$

$$y_{p2}' = -2a \sin 2x + 2b \cos 2x$$

$$y_{p2}'' = -4a \cos 2x - 4b \sin 2x; \text{ zamijenimo sve u } \uparrow \text{ počinjemo}$$

$$\rightarrow -4a \cos 2x - 4b \sin 2x - 4(-2a \sin 2x + 2b \cos 2x) + 8a \cos 2x + 8b \sin 2x = \sin 2x$$

$$\cos 2x (-4a - 8b + 8a) + \sin 2x (-4b + 8a + 8b) = \sin 2x$$

$$\rightarrow \begin{cases} 4a - 8b = 0 \rightarrow a = 2b \\ 8a + 4b = 1 \end{cases} \rightarrow \begin{cases} b = \frac{1}{20} \\ a = \frac{1}{10} \end{cases}$$

$$y_{p2} = \frac{1}{10} \cos 2x + \frac{1}{20} \sin 2x$$

$$y = y_h + y_{p1} + y_{p2} =$$

$$= e^{2x} (C_1 \cos 2x + C_2 \sin 2x) + \frac{e^{2x}}{4} + \frac{1}{10} \cos 2x + \frac{1}{20} \sin 2x$$

Snižavanje reda lin. dif. jne

→ kada su nam poznata part. rj možemo sniziti

$$L(y) = 0 \quad (1)$$

→ neka su $\varphi_1(x), \dots, \varphi_m(x)$ linearno nezavisna rješenja ove jne (1) ($m < n$)

n lin. nez. rješ.

→ tražimo još $n-m$:

$\varphi_{m+1}(x) \dots \varphi_n(x) \rightarrow ?$ rješenja

$\varphi_1(x), \dots, \varphi_m(x), \varphi_{m+1}(x), \dots, \varphi_n(x)$

baza vektorskog prostora

$$P_n(\lambda) = 0 \rightarrow \text{ako znamo u korjenima}$$

$$\frac{P_n(\lambda)}{(\lambda - \lambda_1) \dots (\lambda - \lambda_m)} = Q_{n-m}(\lambda) = 0$$

→ uvodimo supenu:

$$y = \varphi_1 \cdot z ; \quad \boxed{z = z(x)} \rightarrow \text{nova fza}$$

$$y' = \varphi_1' z + \varphi_1 z'$$

$$y'' = \varphi_1'' z + 2\varphi_1' z' + \varphi_1 z''$$

$$y''' = \varphi_1''' z + 3\varphi_1'' z' + 3\varphi_1' z'' + \varphi_1 z'''$$

$$\vdots$$

$$y^{(n)} = \sum \binom{n}{k} \varphi_1^{(k)} \cdot z^{(n-k)}$$

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = 0$$

$$\sum_{k=0}^n \binom{n}{k} \varphi_1^{(k)} \cdot z^{(n-k)} + a_1(x) \sum_{k=0}^{n-1} \binom{n-k}{k} \varphi_1^{(k)} \cdot z^{(n-1-k)} + \dots$$

$$+ \dots + a_n(x) \cdot \varphi_1 z = 0 \rightarrow$$

PASKALOV
TROUGAO

$$\rightarrow z \left(\varphi_1^{(n)} + a_1(x) \cdot \varphi_1^{(n-1)} + \dots + a_n(x) \cdot \varphi_1 \right) + \varphi_1 z^{(n)} + b_1(x) \cdot z^{(n-1)} + \dots + b_{n-1}(x) \cdot z' = 0 \quad /: \varphi_1 \neq 0$$

$L(\varphi_1) = 0$
fer je dio lin. nez. skupa

\rightarrow imamo $z', z'', \dots, z^{(n)}$

\rightarrow uvodimo supenu: $z' = u \rightarrow$

$$\rightarrow u^{(n-1)} + p_1(x) \cdot u^{(n-2)} + \dots + p_{n-1}(x) u = 0$$

znamo $n-m$ rješenja

$$\left(\frac{\varphi_2}{\varphi_1} \right)', \left(\frac{\varphi_3}{\varphi_1} \right)', \dots, \left(\frac{\varphi_m}{\varphi_1} \right)' \rightarrow \text{lin. nez. rješ.}$$

ima izvedeno u knjizi da su nezavisna;
odraditi kući;

$$y = \varphi_1 \cdot z \rightarrow z' = \left(\frac{y}{\varphi_1} \right)' = u$$

ponavlja se
o o o

\rightarrow sve dok ne dobijemo j -nu red $n-m$

$$n=2: y'' + a_1(x) \cdot y' + a_2(x) \cdot y = 0$$

$a_1, a_2 \in C(I); \varphi_1(x)$ rješ;

$\varphi_1(x) \rightarrow$ rješ } linearno nezavis.
 $\varphi_2(x) \rightarrow ?$

$$W(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \varphi_1 \varphi_2' - \varphi_1' \varphi_2 = 0$$

$$\varphi_1 \text{ rješ: } \varphi_1'' + a_1(x) \varphi_1' + a_2(x) \varphi_1 = 0 / \varphi_2$$

$$\varphi_2 \text{ rješ: } \varphi_2'' + a_1(x) \varphi_2' + a_2(x) \varphi_2 = 0 / \varphi_1$$

$$\rightarrow \underbrace{\varphi_1'' - \varphi_2 - \varphi_2'' \varphi_1}_{-W'(x)} + a_1(x) \cdot \underbrace{(\varphi_1' \varphi_2 - \varphi_1 \varphi_2')}_{-W(x)} = 0$$

$$W'(x) = \varphi_1 \varphi_2' + \varphi_1 \varphi_2'' - \varphi_1' \varphi_2' - \varphi_1'' \varphi_2 = \varphi_1 \varphi_2'' - \varphi_1'' \varphi_2$$

$$\boxed{W'(x) + a_1(x) W(x) = 0}$$

$$\frac{dW}{dx} = -a_1(x) dx \rightarrow W(x) = Ce^{-\int a_1(x) dx}$$

$$\varphi_1 \varphi_2' - \varphi_1' \varphi_2 = Ce^{-\int a_1(x) dx}$$

dif. jna po $\varphi_2 \dots$ $\rightarrow = 1$ jer nam treba samo jedno

$$\rightarrow \boxed{y = c_1 \varphi_1 + c_2 \varphi_2} \rightarrow \text{opšte j.}$$

2. način

$$y'' + a_1(x) \cdot y' + a_2(x) \cdot y = 0$$

\rightarrow koef. uz prvi izvod = 0; namještanje

$$y = \varphi \cdot z \rightarrow \text{novi fje}$$

$$y' = \varphi' z + \varphi z'$$

$$y'' = \varphi'' z + 2\varphi' z' + \varphi z''$$

\rightarrow zamijenimo sve u početnu jnu i dobijamo

$$\varphi'' z + 2\varphi' z' + \varphi z'' + a_1(x) \varphi' z + a_1(x) \varphi z' + a_2(x) \varphi z = 0$$

$$\varphi z'' + (2\varphi' + a_1(x) \cdot \varphi) z' + (\varphi'' + a_1(x) \varphi' + a_2(x) \varphi) z = 0$$

\rightarrow postavljamo da je koef. uz 1. izvod = 0

$$\rightarrow \boxed{2\varphi' + a_1(x) \varphi = 0} \rightarrow$$

$$2\varphi' = -a_1(x)\varphi$$

$$\rightarrow \frac{d\varphi}{\varphi} = -\frac{1}{2}a_1(x)dx \rightarrow \varphi = e^{-\int \frac{a_1(x)}{2} dx}$$